Solution of DPP # 8
TARGET : JEE (ADVANCED) 2015
Course: VIJETA & VIJAY (ADP & ADR)

MATHEMATICS

1. $A = \int_0^{505\pi} |\cos x|\,dx = 505\int_0^{\pi} |\cos x|\,dx = 1010$ $\qquad B = \int_{505\pi}^{1007\pi} |\sin x| = (1007 - 505)\int_0^{\pi} |\sin x|\,dx = 1004$

2. $I = \int_0^{100} \sqrt{x} - \left[\sqrt{x}\right] dx = \left(\frac{x^{3/2}}{3/2}\right)_0^{100} - \int_0^{100}\left[\sqrt{x}\right]dx$

$= \frac{2000}{3} - \left[\int_1^4 1.dx + \int_4^9 2.dx + \int_9^{16} 3.dx + \int_{16}^{25} 4.dx + \int_{25}^{36} 5.dx + \int_{36}^{49} 6.dx + \int_{49}^{64} 7.dx + \int_{64}^{81} 8.dx + \int_{81}^{100} 9.dx\right]$

$= \frac{2000}{3} - [3 + 10 + 21 + 36 + 55 + 78 + 105 + 136 + 171] = \frac{2000}{3} - 615 = \frac{155}{3}$

3. $I = \int e^x.\frac{2-x^2}{(1-x)\left(\sqrt{1-x^2}\right)}dx = \int e^x\left(\frac{1}{(1-x)\sqrt{1-x^2}} + \frac{1-x^2}{(1-x)\sqrt{1-x^2}}\right)dx = \int e^x\left(\frac{1}{(1-x)\sqrt{1-x^2}} + \sqrt{\frac{1+x}{1-x}}\right)dx$

$= e^x.\sqrt{\frac{1+x}{1-x}} + c$

4. $f(x) = \int 2x^3.\cos^2 x + 6x^2 \sin x \cos x - 2x^3 \sin^2 x\,dx$

$= \int 2x^3.\cos 2x\,dx + \int \underbrace{3x^2}_{II}\underbrace{\sin 2x\,dx}_{I} = \int x^3.2\cos 2x\,dx + \sin 2x.x^3 - \int x^3.2\cos 2x\,dx$

$\Rightarrow \quad f(x) = x^3 \sin 2x + c \qquad \Rightarrow \quad f(x) = x^3 \sin 2x$

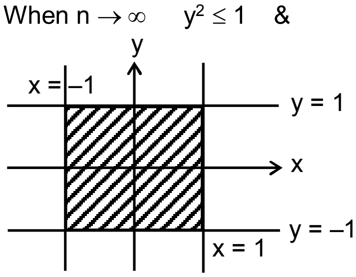
5. $f(x) = f(2 - x)$ & $f(x) = f(4 - x) \Rightarrow f(x) = f(x + 2) \Rightarrow$ f(x) is periodic with period 2

Now $I = \int_0^{50} f(x)\ dx = 25\int_0^2 f(x)\ dx = 75$

6. $I = \int\left(x^6 + x^4 + x^2\right)\sqrt{2x^4 + 3x^2 + 6}\,dx = \int\left(x^5 + x^3 + x\right)\sqrt{2x^6 + 3x^4 + 6x^2}\,dx$

Put $2x^6 + 3x^4 + 6x^2 = t$ $\qquad \therefore \qquad I = \int \sqrt{t}.\frac{dt}{12} = \frac{t^{3/2}}{18} + c$

7. When $n \to \infty$ $\quad y^2 \le 1$ & $x^2 \le 1 \Rightarrow -1 \le y \le 1$ & $-1 \le x \le 1$



$\lim_{n\to\infty} S_n = 4$

8. $I = \int \frac{8x^{43} + 13x^{38}}{\left(x^{13} + x^5 + 1\right)^4}dx = \int \frac{8x^{-9} + 13x^{-14}}{\left(1 + x^{-8} + x^{-13}\right)^4}dx$ $\quad$ Put $\quad 1 + x^{-8} + x^{-13} = t$ $\qquad \therefore \quad I = \int \frac{-dt}{t^4} = \frac{1}{3t^3} + c$

9. $\int e^x (f(x) - f'(x))dx = \phi(x)$ ...(i)

and $\int e^x (f(x) + f'(x))dx = e^x f(x)$...(ii)

equation (i) & (ii)

$2\int e^x f(x)dx = \phi(x) + e^x f(x)$

10. $f'(x) > 0 \Rightarrow f(x)\uparrow \Rightarrow f(x) \geq 1 \ \forall x \geq 1$

$\Rightarrow f'(x) \leq \frac{1}{1+x^2} \ \forall x \geq 1 \Rightarrow \int_1^x f'(x)dx \leq \int_1^x \frac{1}{1+x^2}dx$

$\Rightarrow f(x) - f(1) \leq \tan^{-1}x - \tan^{-1}1 \Rightarrow f(x) \leq 1 - \frac{\pi}{4} + \tan^{-1}x < 1 + \frac{\pi}{4}$

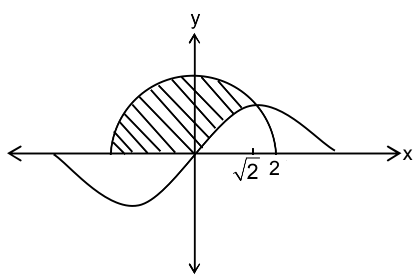
11. Let $\ell = \lim_{x\to 0} \frac{x\int_0^x e^{t^2}dt}{-(e^x - x - 1)}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{x\int_0^x e^{t^2}dt}{-x^2\left(\frac{e^x - x - 1}{x^2}\right)} = -2\lim_{x\to 0} \frac{\int_0^x e^{t^2}dt}{x}\left(\frac{0}{0}\right) = -2\lim_{x\to 0} \frac{e^{x^2}}{1} = -2$

12. We have $\int_0^2 f'(2t)\, e^{f(2t)}dt = 5$ Put $e^{f(2t)} = y \Rightarrow 2f'(2t)\, e^{f(2t)}\, dt = dy$

Now $\frac{1}{2}\int_{e^{f(0)}}^{e^{f(4)}} e^y dy = 5 \Rightarrow \int_{e^{f(0)}}^{e^{f(4)}} e^y dy = 10 \Rightarrow e^{f(4)} - e^{f(0)} = 10 \Rightarrow e^{f(4)} = 10 + e^0 = 11$

Hence $f(4) = \ell n\ 11$

13.



Area to the left of y-axis = π

Area to the right of y-axis = $\int_0^{\sqrt{2}} \left(\sqrt{4 - x^2} - \sqrt{2}\sin\left(\frac{\pi x}{2\sqrt{2}}\right)\right)dx$

$= \left(\frac{x\sqrt{4 - x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right)_0^{\sqrt{2}} + \left(\frac{4}{\pi}\cos\frac{\pi x}{2\sqrt{2}}\right)_0^{\sqrt{2}} = 1 + \pi/2 - 4/\pi$

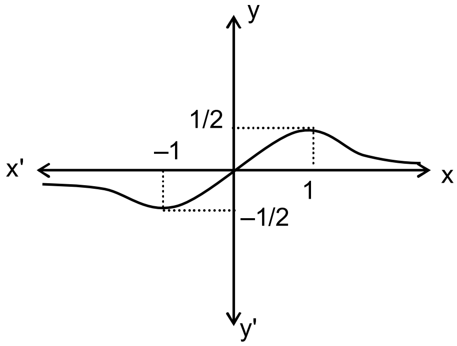
14. $I_1 = \int_{\sin^2 t}^{1+\cos^2 t} xf\left(x(2 - x)\right)dx$

$= \int_{\sin^2 t}^{1+\cos^2 t} \left(1 + \cos^2 t + \sin^2 t - x\right)f\left\{\left(1 + \cos^2 t + \sin^2 t - x\right)\left(2 - \left(1 + \cos^2 t + \sin^2 t - x\right)\right)\right\}$ **[P-5]**

$= 2\int_{\sin^2 t}^{1+\cos^2 t} f\{(2 - x)x\}dx - \int_{\sin^2 t}^{1+\cos^2 t} xf\{(2 - x)x\}dx \Rightarrow I_1 = 2I_2 - I_1 \Rightarrow 2I_1 = 2I_2 \Rightarrow \frac{I_1}{I_2} = 1$

15. Differentiating both sides $g(x) = x - x^2\, g(x)$ $\Rightarrow$ $g(x) = \dfrac{x}{1+x^2}$

Now, graph of y = g(x) is



16. $f(k) = \int_k^{k+1} (x-k)^k\, dx = \left(\dfrac{(x-k)^{k+1}}{k+1}\right)_k^{k+1}$ $\Rightarrow$ $f(k) = \dfrac{1}{k+1}$

Now $\sum_{r=1}^{\infty} (-1)^{r+1} f(r) = f(1) - f(2) + f(3) + \ldots\ldots.$

$= \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{4} - \dfrac{1}{5} + \ldots\ldots = 1 - \ell n\, 2$

17. $f(x + y) = f(x)\, f(y)$ $\Rightarrow$ $f(x) = e^x$

$\therefore$ $g(x) = \dfrac{e^x}{1+e^{2x}} = \dfrac{1}{e^x + e^{-x}}$ $\Rightarrow$ g(x) is an even function

18. $I = \int_{-1}^{0} \dfrac{3\pi}{2} dx + \int_0^2 \dfrac{\pi}{2} dx = \dfrac{5\pi}{2}$ $J = \int_{-2\pi}^{6\pi} \dfrac{\sin x}{|\sin x|} dx + \int_{6\pi}^{7\pi} \dfrac{\sin x}{|\sin x|} dx = 0 + \pi = \pi$

19. $I_{n+1} - I_n = \int_0^{\pi} \dfrac{\sin\left(n + \frac{3}{2}\right)x - \sin\left(n + \frac{1}{2}\right)x}{\sin\frac{x}{2}} dx$

$\Rightarrow$ $I_{n+1} - I_n = \int_0^{\pi} \dfrac{2\cos(n+1)x \quad \sin\frac{x}{2}}{\sin\frac{x}{2}}\ dx$

$\Rightarrow$ $I_{n+1} - I_n = 0$

20. Put $x^5 = t$

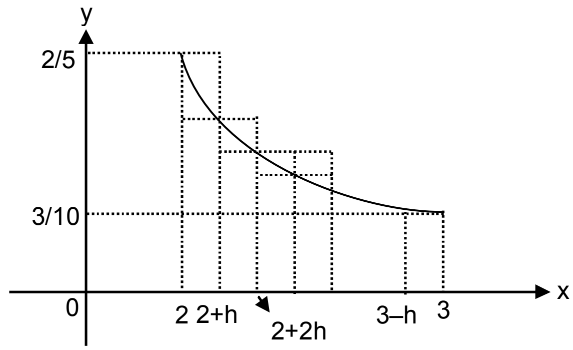
$I = \dfrac{1}{5} \int_0^1 \dfrac{1+t^{2013}}{(1+t)^{2015}}\ dt$

$= \dfrac{1}{5} \int_0^1 \dfrac{1}{(1+t)^{2015}}\ dt + \dfrac{1}{5} \int_0^1 \dfrac{t^{-2}}{\left(t^{-1}+1\right)^{2015}}\ dt$

$= \dfrac{1}{5} \times \dfrac{1}{2014}$

$\therefore$ $p = 5 \times 2014 = 2 \times 5 \times 19 \times 53$

21. Consider $f(x) = \frac{x}{1+x^2}$



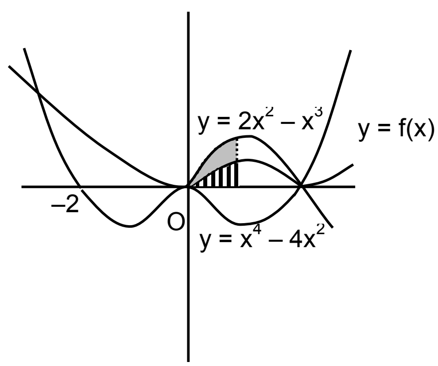
Area bounded by f(x) with x-axis $\int_2^3 \frac{x}{x^2+1} = \ell n\sqrt{2}$

Clearly, $h[f(2)+ f(2 + h) + ... + f(3 – h)] > \ell n\sqrt{2} > h[f(2 + h) + f(2 + 2h) + ... + f(3)]$

22. For all $x \in (0, 1)$

$\Rightarrow \quad \frac{1}{1+x^2} < \frac{1+x^9}{1+x^3} < \frac{1+x^8}{1+x^4} < 1 \qquad \therefore \int_0^1 \frac{1}{1+x^2}dx < I_2 < I_1 < \int_0^1 1 \; dx \qquad \therefore \pi/4 < I_2 < I_1 < 1$

23.



$$\int_0^t \left[f(x) - (x^4 - 4x^2)\right] dx = 2\int_0^t \left[\left(2x^2 - x^3\right) - f(x)\right]dx$$

on differentiating with respect to t.

$\Rightarrow \quad f(t) – (t^4 – 4t^2) = 2(2t^2 – t^3 – f(t)) \qquad \Rightarrow \quad f(t) = \frac{1}{3}(t^4 – 2t^3)$

24. We have $I = \int_2^4 \left(x(3-x)(4+x)(6-x)(10-x) + \sin x \right)dx$ ....(1)

Now $I = \int_2^4 \left((6-x)\left(3-(6-x)\right)\left(4+(6-x)\right)\left(6-(6-x)\right)\left(10-(6-x)\right) + \sin(6-x) \right)dx$

Applying $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \qquad I = \int_2^4 \left((6-x)(x-3)(10-x)x(4+x) + \sin(6-x) \right)dx$ ....(2)

$\therefore$ On adding (1) and (2), we get

$2I = \int_2^4 \left(\sin x + \sin(6-x)\right)dx = \left(-\cos x + \cos(6-x)\right)_2^4 = – \cos 4 + \cos 2 + \cos 2 – \cos 4$

$= 2(\cos 2 – \cos 4)$ Hence $I = \cos 2 – \cos 4$ **Ans.**

25. We have $I = \int_{-\infty}^{a}\left(\frac{\sin^{-1}e^x+\cos^{-1}e^x}{\cot^{-1}e^a+\tan^{-1}e^x}\right)\left(\frac{e^x}{e^{2x}+1}\right)dx = \frac{\pi}{2}\int_{-\infty}^{a}\frac{1}{(\cot^{-1}e^a+\tan^{-1}e^x)}\left(\frac{e^x}{(e^{2x}+1)}\right)dx$

Put $\tan^{-1}e^x = t$ $\Rightarrow$ $\frac{e^x}{e^{2x}+1}dx = dt$

$I = \frac{\pi}{2}\int_{0}^{\tan^{-1}e^a}\frac{dt}{(t+\cot^{-1}e^a)} = \frac{\pi}{2}\left[\ell n(t+\cot^{-1}e^a)\right]_0^{\tan^{-1}e^a} = \frac{\pi}{2}\left[\ell n\left(\frac{\pi}{2}\right)-\ell n\left(\cot^{-1}e^a\right)\right] = -\frac{\pi}{2}\ell n\left(\frac{2}{\pi}\tan^{-1}e^{-a}\right)$

26. We have $I = \int_{k\pi}^{(k+1)\pi}\frac{|\sin 2x|dx}{|\sin x|+|\cos x|}$; put $x = k\pi + t \Rightarrow dx = dt$

$\therefore$ $I = \int_0^{\pi}\frac{|\sin 2x|\,dx}{|\sin x|+|\cos x|} = 2\int_0^{\pi/2}\frac{\sin 2x\,dx}{\sin x+\cos x} = 2\int_0^{\pi/2}\frac{(\sin x+\cos x)^2-1}{\sin x+\cos x}dx$

$= 2\int_0^{\pi/2}(\sin x+\cos x)dx - 2\int_0^{\pi/2}\frac{dx}{\sin x+\cos x} = 4 - 4\int_0^{\pi/4}\frac{dx}{\sin x+\cos x} = 4 - 4J$

27. $f(x) = \int\frac{x^8+4+4x^4-4x^4}{x^4-2x^2+2}\,dx = \int\frac{\left(x^4+2\right)^2-4x^4}{x^4-2x^2+2}dx = \int\frac{(x^4+2x^2+2)(x^4-2x^2+2)}{(x^4-2x^2+2)}\,dx$

$\Rightarrow$ $f(x) = \frac{x^5}{5} + \frac{2x^3}{3} + 2x$

28. $f(x) = \int_0^{\pi/2}\frac{\ell n\left(1+x\sin^2\theta\right)}{\sin^2\theta}d\theta$; $x \ge 0$ $\Rightarrow$ $f'(x) = \int_0^{\pi/2}\frac{1}{1+x\sin^2\theta}\,d\theta$

$\Rightarrow$ $f'(x) = \int_0^{\pi/2}\frac{\sec^2\theta\ \ dq}{1+(1+x)\tan^2\theta}$ put $\tan\theta = t$

$\Rightarrow$ $f'(x) = \int_0^{\infty}\frac{dt}{1+\left\{\left(\sqrt{1+x}\right)t\right\}^2}$ $\Rightarrow$ $f'(x) = \frac{1}{\sqrt{1+x}}\left(\tan^{-1}\left(\sqrt{1+x}\times t\right)\right)_0^{\infty}$

$\Rightarrow$ $f'(x) = \frac{\pi}{2}.\frac{1}{\sqrt{1+x}}$ $\Rightarrow$ $f(x) = \pi\ .\ \sqrt{1+x}+c$ put $x = 0$

$\pi + c = f(0)$ $\Rightarrow$ $c = -\pi$ $\therefore$ $f(x) = \pi\left(\sqrt{1+x}-1\right)$

29. $f(x) = \int_0^x 2t\ \ f(t)dt$ $\Rightarrow$ $f'(x) = 2xf(x)$ $\Rightarrow$ $\frac{f'(x)}{f(x)} = 2x$ $\Rightarrow$ $\ell nf(x) = x^2 + \ell nc$

$\Rightarrow$ $f(x) = c.e^{x^2}$ put $x = 1$ $c.e = f(1) = 0$ $\Rightarrow$ $c = 0$ $\therefore$ $f(x) = 0$

30. $\lim_{n\to\infty}\sum_{r=1}^{n}\left(\left(\frac{3r}{n}\right)^2+2\right)\frac{3}{n} = \int_0^1\left(9x^2+2\right).3dx$

31. $tx = y$ $\Rightarrow$ $\int_0^x f(y)\,dy = xn\,f(x)$ $\Rightarrow$ $f(x) = n[f(x) + xf'(x)]$ $\Rightarrow$ $f(x)(1 - n) = nx\,f'(x)$

$\Rightarrow$ $\frac{f'(x)}{f(x)} = \left(\frac{1-n}{n}\right).\frac{1}{x}$ $\Rightarrow$ $\ell n(f(x)) = \left(\frac{1-n}{n}\right)\ell nx + \ell nc$

$\Rightarrow$ $f(x) = c\,x^{\frac{1-n}{n}}$ as $n \to \infty$ $f(x) = cx^{-1} = \frac{c}{x}$ $\Rightarrow$ $g(x) = \frac{2}{x}$

(32 to 33)

$$I = \int_0^{10\pi} \frac{\cos 4x \cos 5x \cos 6x \cos 7x}{1+e^{2\sin 2x}} dx$$

$$I = \int_0^{10\pi} \frac{\cos 4x \cos 5x \cos 6x \cos 7x}{1+e^{-2\sin 2x}} dx \qquad \text{(from p-5)}$$

$$2I = \int_0^{10\pi} \cos 4x \cos 5x \cos 6x \cos 7x\, dx$$

$$2I = 10\int_0^{\pi} \cos 4x \cos 5x \cos 6x \cos 7x\, dx \qquad \text{(from p-7)}$$

$$2I = 20\int_0^{\pi/2} \cos 4x \cos 5x \cos 6x \cos 7x\, dx \qquad \text{(from p-6)}$$

$$I = 10\int_0^{\pi/2} \cos 4x \cos 5x \cos 6x \cos 7x\, dx \qquad \therefore \quad k = 10$$

Further,

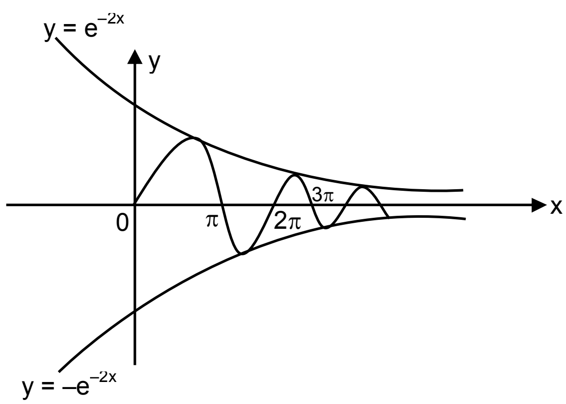
$$I = 5\int_0^{\pi/2} \cos 4x.\cos 6x.(\cos 12x + \cos 2x)$$

$$I = 5\left(\int_0^{\pi/2} \cos 4x \cos 6x \cos 12x\, dx + \int_0^{\pi/2} \cos 2x \cos 4x \cos 6x\, dx\right)$$

$$I = 5\left(0 + 2\int_0^{\pi/4} \cos 2x \cos 4x \cos 6x\, dx\right) \qquad \text{(from p-6)}$$

$$I = 10\int_0^{\pi/4} \cos 2x \cos 4x \cos 6x\, dx \qquad \therefore \quad \lambda = 10$$

(34 to 36)



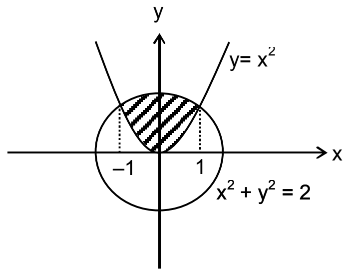
Now $S_i = \left| \int_{i\pi}^{(i+1)\pi} e^{-2x}\sin x \; dx \right| \Rightarrow \quad S_i = \left| \left(\frac{e^{-2x}}{5}(-2\sin x - \cos x) \right)_{i\pi}^{(i+1)\pi} \right|$

$\Rightarrow \quad S_i = \frac{1}{5} \left| e^{-2(i+1)\pi}\cos\ (i+1)\pi - e^{-2i\pi}\cos i\pi \right| \quad \Rightarrow \quad S_i = \frac{e^{-2i\pi}}{5}\left(1+e^{-2\pi}\right)$

(i) $S_0 = \frac{1+e^{-2\pi}}{5}$ (ii) $\frac{S_{2014}}{S_{2015}} = e^{2\pi}$ (ii) $\sum_{i=0}^{\infty} S_i = \frac{\frac{1+e^{-2\pi}}{5}}{1-e^{-2\pi}} = \frac{e^{2\pi}+1}{5\left(e^{2\pi}-1\right)}$

37. $f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}$ Differentiable both side w.r.t. y $f'\left(\frac{x}{y}\right).\left(\frac{-x}{y^2}\right)=\frac{-f(x)}{f^2(y)}.f'(y)$

Put y = 1 $\Rightarrow$ $f'(x).x = 2f(x)$ $\Rightarrow$ $f(x) = x^2$



$$A = 2\int_0^1\left(\sqrt{2-x^2}-x^2\right)\ dx=\frac{1}{3}+\frac{\pi}{2} \quad \Rightarrow \quad 2A = \frac{2}{3}+\pi \quad \Rightarrow \quad [2A] = 3$$

38. Let $I = \int_{\pi/4}^{\pi/2}\frac{dx}{\cos x(\tan x+1+2\sqrt{\tan x})\sqrt{\tan x\cos^2 x}}$ $I = \int_{\pi/4}^{\pi/2}\frac{\sec^2 x\ dx}{(1+\sqrt{\tan x})^2\sqrt{\tan x}}$

Put $\tan x = t^2$ $\Rightarrow$ $\sec^2 x\ dx = 2t\ dt$ $I = \int_1^{\infty}\frac{2t\,dt}{(t+1)^2\cdot t}$ $I = -2\left[\frac{1}{t+1}\right]_1^{\infty} = -2\left[0-\frac{1}{2}\right] = 1$

39. We have $f(2x) = 3 f(x)$ ....(1) and $\int_0^1 f(x)dx=1$ ....(2)

From (1) and (2), $\frac{1}{3}\int_0^1 f(2x)dx=1$

Put 2x = t, $\frac{1}{6}\int_0^2 f(t)dt=1$ $\Rightarrow$ $\int_0^2 f(t)dt=6$ $\Rightarrow$ $\int_0^1 f(t)dt+\int_1^2 f(t)dt=6$

Hence $\int_1^2 f(t)dt=6-\int_0^1 f(t)dt = 6 - 1 = 5$

40. Consider $I_2 = \int_0^1 x^{1004}\left(1-x^{2010}\right)^{1004}dx$ Put $x^{1005} = t$ $\Rightarrow$ $1005x^{1004}dx = dt$

So $I_2 = \frac{1}{1005}\int_0^1\left(1-t^2\right)^{1004}dt$ …(i) Also $I_2 = \frac{1}{1005}\int_0^1\left[1-(1-t)^2\right]^{1004}dt$ …(ii)

$\Rightarrow$ $I_2 = \frac{1}{1005}\int_0^1\left(t(2-t)\right)^{1004}dt = \frac{1}{1005}\int_0^1 t^{1004}(2-t)^{1004}dx$ Put t = 2y $\Rightarrow$ dt = 2dy

So $I_2 = \frac{1}{1005}\int_0^{1/2}(2y)^{1004}(2-2y)^{1004}2dy = \frac{1}{1005}2.2^{1004}.2^{1004}\int_0^{1/2}y^{1004}(1-y)^{1004}dy$

$I_2 = \frac{1}{1005}2^{2009}\int_0^{1/2}y^{1004}(1-y)^{1004}dy$ …(iii)

Now $I_1 = \int_0^1 x^{1004}(1-x)^{1004}dx = 2\int_0^{1/2}x^{1004}(1-x)^{1004}dx$ …(iv)

$\therefore$ From (iii) and (iv) we get

$I_2 = \frac{1}{1005}2^{2010}\frac{I_1}{4}$ $\Rightarrow$ $2^{2010}\frac{I_1}{I_2} = 4020$